Name: ________________ **ID (helpful but not necessary):** ________________

Clear your desk of everything except pens, pencils and erasers. Show all work clearly and in order. No notes, phones and calculators. You have 50 minutes to finish the 6 pages for 100 points.

1. (20 points) Check that whether the following statements are True or False. (a-d 2 points; e-h 3 points).

2 **(a) True or False.** Let R^+ denote the set of positive real numbers. Define the operation of addition, denoted $\oplus$, by

$$x \oplus y = x \cdot y, \quad \text{for all } x, y \in R^+.$$

HW Sec3.1, 12

Then the scalar 1 is the **zero vector** satisfying Axiom A.3 in R^+ (with respect to $\oplus$).

2 **(b) True or False.** Let R^+ denote the set of positive real numbers. Define the operation of scalar multiplication, denoted $\circ$, by

$$\alpha \circ x = x^\alpha, \quad \text{for all } x \in R^+.$$

HW. Sec3.1, 12

Then for two scalars α, β, we have

$$(\alpha\beta) \circ x = \alpha \circ (\beta \circ x)$$

HW. Sec3.2.4(c)

2 **(c) True or False.** The vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is in the **null space** of the matrix $\begin{bmatrix} 1 & 3 & -4 \\ 2 & -1 & -1 \\ -1 & -3 & 4 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

2 **(d) True or False.** The two vectors $\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \end{bmatrix} \right\}$ form a basis for $\mathbb{R}^2$. HW. Sec3.3.1(a).

3 **(e) True or False.** If -1 is an eigenvalue of an $n \times n$ matrix A, then 1 is an eigenvalue of A^2.

optional HW. Sec 6.1. 6*

3 **(f) True or False.** The matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ has two eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -1$.

Lecture Notes $p(\lambda) = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0$ $\lambda_1 = i, \lambda_2 = -i$. No real eigenvalues.

3 **(g) True or False.** Consider the mapping L from $\mathbb{R}^3$ to $\mathbb{R}^2$ given by

$$L(\mathbf{x}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \mathbf{x} \in \mathbb{R}^3$$

L is NOT a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^2$. HW. Sec 4.2. 5(b).

3 **(h) True or False.** Consider the mapping M from $\mathbb{R}^2$ to $\mathbb{R}^1$ defined by

$$M(\mathbf{x}) = (x_1^2 + x_2^2)^{1/2}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$$

M is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^1$. Lecture Notes. (Textbook example)

$M(\alpha x) = |\alpha| \cdot M(x)$.

2. (20 points) Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & -3 & -1 \\ -2 & -4 & 6 & 3 \end{bmatrix}$$

(a)(12 points) Find the null space of the matrix A. Show your work. H.W. Sec 3.2. 4(b)

$A\bar{x} = \bar{0}$

$$\left[\begin{array}{cccc|c} 1 & 2 & -3 & -1 & 0 \\ -2 & -4 & 6 & 3 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 2 & -3 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|c} 1 & 2 & -3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \quad 6pt$$

$x_1 + 2x_2 - 3x_3 = 0$
$x_4 = 0$

$\Rightarrow$ $\boxed{x_2 = \alpha,\ x_3 = \beta.\ x_1 = -2\alpha + 3\beta,\ x_4 = 0}$ 2pt

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -2\alpha+3\beta \\ \alpha \\ \beta \\ 0 \end{pmatrix} = \alpha \cdot \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \beta \cdot \begin{pmatrix} 3 \\ 0 \\ 1 \\ 0 \end{pmatrix},\quad N(A) = \underbrace{\left\{ \alpha \cdot \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \beta \cdot \begin{pmatrix} 3 \\ 0 \\ 1 \\ 0 \end{pmatrix} \,\middle|\, \alpha, \beta \in \mathbb{R} \right\}}_{2pt}.$$

2pt

(b)(4 points) Find a basis for the null space of A.

$\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ form a basis for the null of A.

2pt 2pt

(c)(4 points) Find the rank of the matrix A and the dimension of the null space of A.

rank$(A) = 2$. 2pt

dimension of the null space $= 2$ 2pt

3. (20 points) Let

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} -1 \\ 3 \\ 4 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 6 \\ 2 \\ 4 \end{bmatrix}$$

HW. Sec3.4. 5.

(a)(8 points) Show that $\mathbf{x}_1$ and $\mathbf{x}_2$, are linearly independent.

Let $c_1\bar{x}_1 + c_2\bar{x}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \iff \begin{bmatrix} 1 & -1 \\ 2 & 3 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. 1pt

2pt

Augmented matrix. $\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 2 & 3 & 0 \\ 3 & 4 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 5 & 0 \\ 0 & 7 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$ 2pt

$\begin{cases} c_1 - c_2 = 0 \\ c_2 = 0 \end{cases} \Rightarrow c_1 = c_2 = 0$ 2pt

Therefore, $\bar{x}_1, \bar{x}_2$ are linearly independent 1pt

(b)(10 points) Express $\mathbf{x}_3$ as a linear combination of $\mathbf{x}_1, \mathbf{x}_2$. (if possible.) (Review session).

Let $c_1\bar{x}_1 + c_2\bar{x}_2 = \bar{x}_3 \iff \begin{bmatrix} 1 & -1 \\ 2 & 3 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 4 \end{bmatrix}$ 2pt

2pt

Augmented matrix $\left[\begin{array}{cc|c} 1 & -1 & 6 \\ 2 & 3 & 2 \\ 3 & 4 & 4 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -1 & 6 \\ 0 & 5 & -10 \\ 0 & 7 & -14 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -1 & 6 \\ 0 & 1 & -2 \\ 0 & 1 & -2 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{array}\right]$

3pt

1p 1p

$c_1 = 4, \quad c_2 = -2.$

$\bar{x}_3 = 4\bar{x}_1 - 2\bar{x}_2$ 1pt

(c)(2 points) Find the dimension of Span $(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$. (No explain needed.)

dimension = 2. 2pt

4. (20 points) Let

$$A = \begin{bmatrix} 2 & -8 \\ 1 & -4 \end{bmatrix}$$

HW Sec 6.3. 1.(c).

(a)(12 points) Find the eigenvalues of A and one eigenvector corresponding to each eigenvalue.

$p(\lambda) = \det(A - \lambda \cdot I) = \begin{vmatrix} 2-\lambda & -8 \\ 1 & -4-\lambda \end{vmatrix} = (2-\lambda)\cdot(-4-\lambda) - (-8)\cdot 1 = \lambda^2 + 2\lambda - 8 + 8 = \lambda^2 + 2\lambda = 0$

2pt

$\lambda_1 = 0, \quad \lambda_2 = -2.$

2pt 2pt

$\lambda_1 = 0$

$A - 0\cdot I = \begin{bmatrix} 2 & -8 \\ 1 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & -4 \\ 0 & 0 \end{bmatrix}, \quad x_1 - 4x_2 = 0, \quad x_1 = 4x_2$

eigenvector $v_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ 3pt

$\lambda_2 = -2$

$A + 2\cdot I = \begin{bmatrix} 4 & -8 \\ 1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}, \quad x_1 - 2x_2 = 0, \quad x_1 = 2x_2$

eigenvector $v_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ 3pt.

(b)(6 points) Find a nonsingular matrix X and its inverse X^{-1} such that the matrix A can be factorized into a product XDX^{-1}, where D is diagonal.

$X = (v_1, v_2) = \begin{pmatrix} 4 & 2 \\ 1 & 1 \end{pmatrix}.$ 2pt

$X^{-1} = \frac{1}{4-2}\begin{bmatrix} 1 & -2 \\ -1 & 4 \end{bmatrix}$

$= \frac{1}{2}\begin{bmatrix} 1 & -2 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -1 \\ -\frac{1}{2} & 2 \end{bmatrix}$ 2pt

$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix}$ 2pt

Then $A = X \cdot D \cdot X^{-1}$

$= \begin{bmatrix} 4 & 2 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{2} & -1 \\ -\frac{1}{2} & 2 \end{bmatrix}$

(c)(2 points) Use your answer to part (b) to find the eigenvalues of A^5. (No work needed).

(optional HW. Sec 6.1, 6*)

A has eigenvalues $\lambda_1 = 0, \quad \lambda_2 = -2$

Then A^5 has eigenvalues $0^5 = 0$ and $(-2)^5 = -32$.

1pt 1pt

5. (10 points) For $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3$, let

HW. Sec4.2. 2 (a).

$$L(\mathbf{x}) = \begin{bmatrix} x_1 + x_2 \\ 0 \end{bmatrix}$$

(a)(2 points) Is it TRUE or FALSE that for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$ and scalar α,

$$L(\mathbf{x} + \mathbf{y}) = L(\mathbf{x}) + L(\mathbf{y}), \quad L(\alpha \mathbf{x}) = \alpha L(\mathbf{x})$$

True. 2pt

(b)(6 points) Find a matrix A such that $L(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^3$.

$\bar{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\bar{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\bar{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$L(\bar{e}_1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $L(\bar{e}_2) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $L(\bar{e}_3) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Matrix representation: $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

2pt 2pt 2pt

(c)(2 points) Let S be a subspace in $\mathbb{R}^3$ given by

$$S = \{ \ \alpha \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \ | \ \alpha \in \mathbb{R} \ \}$$

Find $L(S)$, i.e., the image of S.

If $\bar{x} = \alpha \cdot \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha \\ -\alpha \\ 0 \end{pmatrix}$, then $L(\bar{x}) = \begin{bmatrix} \alpha - \alpha \\ 0 \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ 2pt

Therefore, $L(S) = \{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \}$

6. (10 points) Let P_3 be the vector space of all polynomials of degree less than 3. Consider

$$p(x) = x + 2, \quad q(x) = x^2 - 1$$

optional H/W
Sec 3.2. 19. (d).

(a)(4 points) Show that $p(x)$ and $q(x)$ are linearly independent in P_3

Let $c_1(x+2) + c_2(x^2-1) = 0$ 1pt

$\Leftrightarrow 2c_1 - c_2 + c_1 x + c_2 x^2 = 0$ 1pt

$\Rightarrow 2c_1 - c_2 = 0, \quad c_1 = 0, \quad c_2 = 0$

1pt 1pt

$\Rightarrow c_1 = c_2 = 0$

Therefore, $p(x) = x+2$ and $q(x) = x^2-1$ are linearly independent.

(b)(4 points) Let $h(x) = 1 + x$. Is $h(x)$ in Span $(x + 2, x^2 - 1)$?

Let $c_1(x+2) + c_2(x^2-1) = 1 + x$. 1pt

$\Leftrightarrow \underline{2c_1 - c_2} + c_1 x + c_2 x^2 = 1 + x$ 1pt

$\Rightarrow 2c_1 - c_2 = 1, \quad c_1 = 1, \quad c_2 = 0$ 1pt

$\Rightarrow 2c_1 = 1$ and $c_1 = 1$. contradiction. The system is inconsistent.

There is no c_1, c_2 such that $h(x) = c_1 \cdot p(x) + c_2 q(x)$

i.e. $h(x)$ is not in 1pt Span$(p(x), q(x))$

(c)(2 points) Is $\{x + 2, x^2 - 1\}$ a spanning set for P_3?

2pt

By (b). $\{x+2, x^2-1\}$ is NOT a spanning set for P_3.

(Because $h(x) = 1+x$ in P_3 cannot be expressed as a linear combination of $x+2$ and x^2-1.